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DIMENSION AND ENTROPY OF REGULAR CURVES

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§1 - INTRODUCTION

There does not exist a unique definition of the dimension. Each definition has to be adapted to the object one wants to study. This could be one of the many lessons, Mandelbrot has taught us. The topological dimension, the Hausdorff dimension, the Kolmogoroff and Bouligand dimension, etc..., all have different definitions. Their value may coincide in special cases, but in general, they do not.

My topic deals with the dimension attached to infinite plane rectifiable curves. "Fractal" curves are thus excluded. I will also introduce the concept of entropy and I will show that the entropy increases with the dimension. Thus the dimension measures the complexity of the curve. This last statement is another general idea which is beautifully illustrated in the books of Mandelbrot [3].

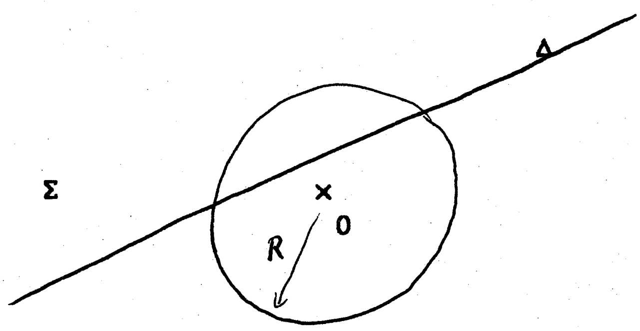
§2 - THE DIMENSION OF A CURVE

Let Δ be an infinite straight line. The disc D_R centered on 0 with radius R intersects Δ. As R goes to infinity

$$\text{length } (\Delta \cap D_R) \sim 2R$$

Let Σ be the half plane above Δ :

$$\text{area } (\Sigma \cap D_R) \sim \frac{\pi}{2} R^2$$



The exponent of R in the two formulas reflects the dimension of both Δ and Σ. This simple observation leads us to the following definition.

Let Γ be an infinite rectifiable curve the length of which is finite in every bounded region of the plane. (Such a curve tends to infinity). By definition

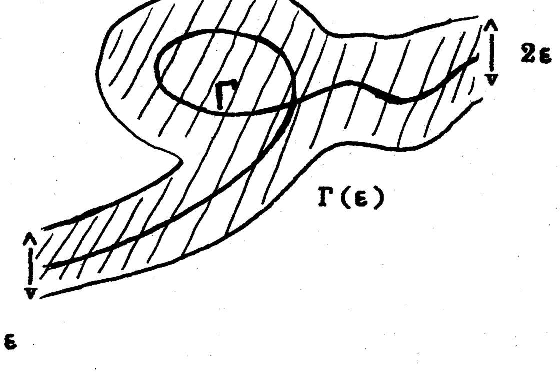
$$\dim(\Gamma) = \lim_{R\to\infty} \frac{\log \text{length}\,(\Gamma \cap D_R)}{\log R}$$

Actually this definition may well be criticized. The limit may not exist. If such is the case, we would the consider

$$\left\{\begin{aligned} \overline{\dim}\,(\Gamma) &= \limsup_{R\to\infty} \frac{\log \text{length}\,(\Gamma \cap D_R)}{\log R} \\ \underline{\dim}\,(\Gamma) &= \liminf_{R\to\infty} \frac{\log \text{length}\,(\Gamma \cap D_R)}{\log R} \end{aligned}\right. \tag{1}$$

Now, if the curve is extremely slow to approach the point at infinity, both ratios may be larger than 2, the dimension of the plane. This shows that our definition is unreasonable. We thus must modify our definition.

Let $\varepsilon > 0$ and let $\Gamma(\varepsilon)$ be the ε-magnification of Γ



$$\left\{\begin{aligned} \overline{\dim}\,(\Gamma) &= \lim_{\varepsilon\to 0}\limsup_{R\to\infty} \frac{\log \text{area}\,(\Gamma(\varepsilon) \cap R)}{\log R} \\ \underline{\dim}\,(\Gamma) &= \lim_{\varepsilon\to 0}\liminf_{R\to\infty} \frac{\log \text{area}\,(\Gamma(\varepsilon) \cap R)}{\log R} \end{aligned}\right. \tag{2}$$

Then quite obviously

$$1 \leq \underline{\dim}\,(\Gamma) \leq \overline{\dim}\,(\Gamma) \leq 2.$$

Furthermore, for all α and β such that $1 \leq \alpha \leq \beta \leq 2$ there exists a curve Γ such that

$$\underline{\dim}\,(\Gamma) = \alpha \quad \text{and} \quad \overline{\dim}\,(\Gamma) = \beta.$$

Examples : Let $\alpha > 0$. The spiral $\rho = \alpha$ has dimension

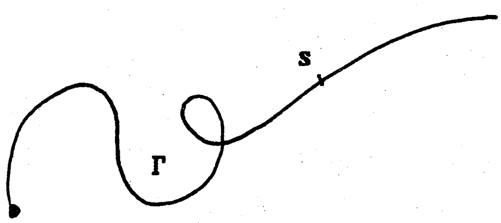
$$\min\left\{2, \frac{1}{1+\alpha}\right\}$$

and the spiral $\rho = \exp \theta$ has dimension 1.

For further examples see Mendes France and Tenenbaum [4] and Dekking and Mendes France [1].

§3 - RESOLVABLE CURVES

Let $s > 0$ and let Γ_s be the starting portion of Γ with length s.



We denote by $\Gamma_s(\varepsilon)$ the ε-magnification of Γ.

We say that Γ is ***resolvable*** if there exists a positive ε such that

$$\liminf_{s \to \infty} \frac{\text{area } \Gamma_s(\varepsilon)}{s} > 0$$

The inequality then stands for all positive ε. A resolvable curve cannot "bunch up" too much.

If $0 < \alpha < 1$ the spiral $\rho = \theta^{\alpha}$ is not resolvable whereas for $\alpha > 1$ it is resolvable.

If Γ is resolvable, then for all $\varepsilon>0$

$$C_1(\varepsilon)\ \text{length}\ (\Gamma\cap R) \leq \text{area}\ (\Gamma(\varepsilon)\cap R) \leq C_2(\varepsilon)\ \text{length}\ (\Gamma\cap R)$$

where $0 < C_1(\varepsilon) < C_2(\varepsilon)$, hence its dimension is given by formula (1).

$$\begin{cases} \overline{\dim}\ \Gamma = \limsup\limits_{R\to\infty} \dfrac{\log \text{length}\ (\Gamma\cap D_R)}{\log R} \\[2ex] \underline{\dim}\ \Gamma = \liminf\limits_{R\to\infty} \dfrac{\log \text{length}\ (\Gamma\cap D_R)}{\log R} \end{cases}$$

If Γ is not resolvable, we only have an inequality

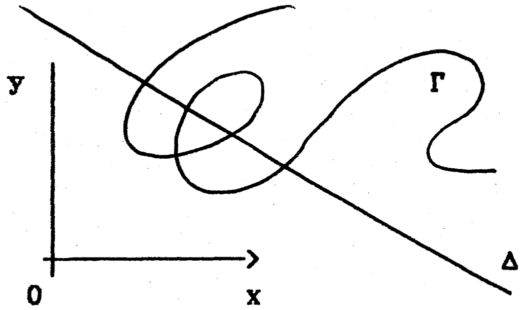
$$\begin{cases} \overline{\dim}\ \Gamma \leq \limsup\limits_{R\to\infty} \dfrac{\log \text{length}\ (\Gamma\cap D_R)}{\log R} \\[2ex] \underline{\dim}\ \Gamma \leq \liminf\limits_{R\to\infty} \dfrac{\log \text{length}\ (\Gamma\cap D_R)}{\log R} \end{cases}$$

Nonresolvable curves are thus more complicated to study. They are in fact more complex as will now be shown with the help of entropy.

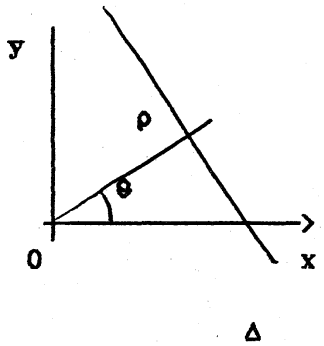
§4 - ENTROPY OF FINITE CURVES

In this paragraph we shall define the entropy of finite curves. It is only in paragraph 6 that we shall extend the definition to infinite curves.

Let $\Omega = \Omega(\Gamma)$ be the set of straight lines which intersect Γ.



A straight line is determined by the two parameters θ and ρ.



We identify the two couples $(\rho,\theta+\pi)$ and $(-\rho,\theta)$ thus generating a Möbius manifold. A straight line Δ in the x,y plane is hence represented as a point on the Möbius manifold and conversely.

The set Ω of straight lines which intersect Γ is represented by a set Ω (we keep the same notation) of points on the manifold. Let p be the uniform normalized measure on Ω. Ω is thus endowed with a probability measure p ; $p(\Omega)=1$.

Let p_n be the probability that a line Δ intersects Γ in exactly n points $(n \geq 1)$:

$$p_n = p \{\Delta \in \Omega \,/\, \text{card } (\Delta \cap \Gamma) = n\}.$$

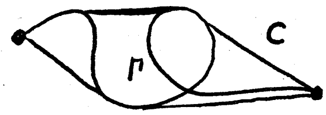
Hence

$$\sum_1^{\infty} p_n = 1$$

A remarkable result of Steinhaus [5],[6] states that the expectation of the number of intersection points is

$$2 \text{ length } (\Gamma) \;/\; C$$

where C (the perimeter of Γ) is the length of the boundary of the convex hull of Γ.



In other words

$$\sum_{n=1}^{\infty} np_n = 2 \text{ length } (\Gamma) \;/\; C \,.$$

Note that if Γ is a finite straight segment then

$$2 \text{ length } (\Gamma) = C \quad ;$$

for all other curves

$$2 \text{ length } (\Gamma) > C,$$

an observation which we shall use later.

The entropy S(Γ) of Γ is by definition

$$S(\Gamma) = \sum_{n=1}^{\infty} p_n \log \frac{1}{p_n}$$

where as usual $0.\infty=0$. If Γ is a finite straight segment then obviously $S(\Gamma)=0$. Small entropy means simple curves and large entropy means complex curves.

Theorem 1 -

$$S(\Gamma) \leq \log \frac{2 \text{ length } (\Gamma)}{C} + \frac{\beta}{e^{\beta}-1}$$

where

$$\beta = \log \frac{2 \text{ length } (\Gamma)}{2 \text{ length } (\Gamma)-C} > 0$$

We shall prove this result but before notice that $\beta/(e^{\beta}-1)$ is comprised between 0 and 1 and hence plays no important role. We should think of $S(\Gamma)$ as the metric entropy of Γ and $\log \dfrac{2 \text{ length } (\Gamma)}{C}$ as the topological entropy $S_T(\Gamma)$.

Proof - We maximize

$$\sum_{1}^{\infty} p_n \log \frac{1}{p_n}$$

with the two contraints

$$\sum_{1}^{\infty} p_n = 1 \quad \text{and} \quad \sum_{1}^{\infty} n p_n = 2 \text{ length } (\Gamma)/C.$$

The Lagrange technique introduces the auxiliary function

$$U = \sum_{1}^{\infty} p_n \log \frac{1}{p_n} - \alpha \sum_{1}^{\infty} p_n - \beta \sum_{1}^{\infty} n p_n$$

where α and β are two unknown constants.

Solving

$$\frac{\partial U}{\partial p_n} = 0$$

we obtain

$$p_n = a\, e^{-\beta n}$$

where a and β are two constants which are determined by the two contraints. We obtain

$$a = e^{\beta}-1$$

and

$$\beta = \log \frac{2 \text{ length } (\Gamma)}{2 \text{ length } (\Gamma)-C}$$

Hence

$$p_n = \frac{C}{2 \text{ length } (\Gamma)-C} \left(1 - \frac{C}{2 \text{ length } (\Gamma)}\right)^n$$

and

$$S_{Max} = S_T = \log \frac{2\ \mathbf{length}\ (\Gamma)}{C} + \frac{\beta}{e^{\beta}-1}. \qquad \text{QED}$$

§5 - THE TEMPERATURE OF A CURVE

Before considering infinite curves we wish to comment on the previous computation. When physicists want to define equilibrium of a gas, they are led to a similar extremal problem in which their parameter β is identified with the inverse temperature of the gas (actually $\beta=1/kT$ where **k** is the Boltzmann constant which fixes the scale of temperature). Choosing **k**=1, and mimicking physics, we define our β as being inverse temperature of the curve Γ :

$$T = \left[\log \frac{2\ \mathbf{length}\ (\Gamma)}{2\ \mathbf{length}\ (\Gamma)-C}\right]^{-1} \qquad (4)$$

Note that **T>0** as a temperature should be. At this point we identify the length of Γ with its volume **V**. The pressure **P** is chosen to be **1/C** (the higher the pressure, the smaller **C** and the more Γ is confined in a small region). Our formula (4) now reads

$$T = \left[\log \frac{2V}{2V - \frac{1}{P}}\right]^{-1}$$

or

$$PV = \frac{1}{2} \frac{1}{1 - \exp\left(-\frac{1}{T}\right)} .$$

When T decreases to zero, **PV** tends to **1/2**. But the equality **PV = 1/2** is equivalent to **length** **(Γ)/C = 1/2**, hence Γ is a finite segment, the entropy of which is **0**. This is known as Nernst law. At zero temperature all curves freeze to straight segments !

We now let T increare to infinity :

$$1 - \exp\left(-\frac{1}{T}\right) \sim \frac{1}{T}$$

so

$$PV \sim \frac{1}{2} T.$$

We recognize the Boyle-Mariotte law which in our context shows that at high temperatures all curves behave like perfect gases !

For further discussion see Dupain, Kamae, Mendes France [2].

§6 - ENTROPY OF AN INFINITE CURVE

We close the parenthesis concerning elementary thermodynamics and go back to the definition of the entropy.

Let Γ be an infinite curve and let Γ_s be a finite section of Γ with **length s**. The entropy of Γ_s is $S(\Gamma_s)$ which was defined in paragraph 4.

It is convenient to normalize $S(\Gamma_s)$ and to consider the ratio

$$S(\Gamma_s)/\log s.$$

By definition, the upper (resp. lower) entropy of Γ is

$$\begin{cases} \overline{h}\,(\Gamma) = \limsup\limits_{s\to\infty} \dfrac{S(\Gamma_s)}{\log s} \\ \underline{h}\,(\Gamma) = \liminf\limits_{s\to\infty} \dfrac{S(\Gamma_s)}{\log s} \end{cases}$$

We consider also the topological entropies

$$\begin{cases} \overline{H}\,(\Gamma) = \limsup\limits_{s\to\infty} \dfrac{S_T(\Gamma_s)}{\log s} \\ \underline{H}\,(\Gamma) = \liminf\limits_{s\to\infty} \dfrac{S_T(\Gamma_s)}{\log s} \end{cases}$$

Our theorem implies

$$0 \leqslant \underline{h}(\Gamma) \leqslant \overline{h}(\Gamma) \leqslant \overline{H}(\Gamma) \leqslant 1.$$

Furthermore, for all α,β $0 \leq \alpha \leq \beta \leq 1$ there exists a Γ such that

$$\underline{h}(\Gamma) = \alpha \quad \text{and} \quad \overline{h}(\Gamma) = \beta.$$

For example, the spiral $\rho = \theta^{\kappa}$ has entropy

$$h(\Gamma) = \frac{1}{1+\alpha}.$$

The spiral $\rho = \exp \theta$ has entropy 0 and $\rho = \log \theta$ has entropy 1.

A zero entropy curve should be considered as deterministic and a one entropy curve should be considered as chaotic ($\rho = \exp \theta$ corresponds to biological growth, maximal order, whereas $\rho = \log \theta$ may be linked with the shape of whirls like in a turbulent *flow*...)

Theorem 2 - *For all infinite curves* Γ

$$1 - \frac{1}{\overline{\dim}\,(\Gamma)} \leq \overline{H}(\Gamma)$$

$$1 - \frac{1}{\underline{\dim}\,(\Gamma)} \leq \underline{H}(\Gamma)$$

If Γ *is resolvable*

$$\overline{h}(\Gamma) \leq 1 - \frac{1}{\overline{\dim}\,(\Gamma)} = \overline{H}(\Gamma)$$

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The theorem underlines the fact that increasing the entropy will (often) augment the dimension. Were our inequalities equalities, the above statement would be made more precise.

Proof - Let C_s denote the permeter of Γ_s.

$$\overline{\dim}\,(\Gamma) \leq \limsup_{R\to\infty} \frac{\log \text{length}\,(\Gamma \cap D_R)}{\log R} = \limsup_{s\to\infty} \frac{\log s}{\log C_s}.$$

Now

$$\overline{H}(\Gamma) = \limsup_{s\to\infty} \frac{\log 2s/C_s}{\log 2s} = 1 - \frac{1}{\limsup\limits_{s\to\infty} \dfrac{\log s}{\log C_s}} \geq 1 - \frac{1}{\overline{\dim}\,(\Gamma)} .$$

The same calculation holds with the lower limit.

Suppose now Γ is resolvable. Then

$$\overline{h}(\Gamma) \leq \overline{H}(\Gamma) = 1 - \frac{1}{\limsup\limits_{s\to\infty} \dfrac{\log s}{\log C_s}} = 1 - \frac{1}{\overline{\dim}\,(\Gamma)}$$

and similarity

$$\underline{h}(\Gamma) \leq 1 - \frac{1}{\underline{\dim}\,(\Gamma)} . \qquad \text{QED}$$

Corollary 1 - ***All one dimensional (resolvable) curves are deterministic.***

Indeed, if $\overline{\dim}\,(\Gamma) = 1$ then $\overline{h}\,(\Gamma) = 0$.

Corollary 2 - ***If a curve*** Γ ***has upper entropy*** $\overline{h}\,(\Gamma)$ ***strictly larger than*** $1/2$, ***then*** Γ ***is nonresolvble.***

Indeed, if Γ were resolvable, then

$$\frac{1}{2} < \overline{h}\,(\Gamma) \leq 1 - \frac{1}{\overline{\dim}\,(\Gamma)}$$

hence $\overline{\dim}\,(\Gamma) > 2$. Absurd.

This last corollary will serve us as a conclusion. To say that $\overline{h}(\Gamma) > 1/2$ is to say that Γ is chaotic (or at least half way towards chaos). Non resolvability expresses the fact that the curve is "pinched" onto itself. One is never sure on which branch of the curve one is. Our corollary thus states that "chaos brings confusion".

A very banal statement indeed !

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